A boundary layer theorem, with applications to rotating cylinders

By M. B. GLAUERT
Department of Mathematics, University of Manchester

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SUMMARY

If, in a given solution of the boundary layer equations, the position of the wall is varied, then additional solutions of the boundary layer equations may be deduced. The theorem considers the nature of such solutions, for the general case of time-dependent three-dimensional compressible flow.

Applications of the theorem arise in several different fields, and it is shown that useful quantitative results can often be obtained with the minimum of calculation. In this paper, chief attention is focused on the case of a rotating circular cylinder, and explicit formulae are developed for the skin friction, valid for sufficiently low rotational speeds. The important results which the theorem gives for slip flow have been noted by previous authors, and only a brief discussion is given here, but certain extensions to these previous treatments are made. Other applications of the theorem are briefly mentioned.

1. INTRODUCTION

The theorem concerns the additional solutions of the boundary layer equations that may be deduced from a previously known solution, by considering the body surface to be at $z = \zeta(x, y, t)$ instead of at $z = 0$, where (x, y) are orthogonal curvilinear coordinates on the body surface and t is the time. For the case of steady two-dimensional incompressible flow, the theorem was given by Prandtl (1938), though he does not appear to have made any applications of the result. More recently Nonweiler (1952) effectively uses a limited form of the theorem for steady two-dimensional compressible flow, in a paper concerned with the effect of slip in a laminar boundary layer. He introduces the concept of a 'plane of zero-slip', at a small distance below the actual surface, and applies the usual no-slip boundary conditions there. He deduces results equivalent to those following from the theorem, though his treatment is somewhat imprecise. Mangler (1956) has attempted to extend Nonweiler's work to steady three-dimensional flow by employing a three-dimensional form of the theorem, but the boundary layer equations he uses are inadequate, as all curvature terms are omitted.

In this paper the theorem is proved for the general case of time-dependent three-dimensional compressible flow. The form of the new surface $z = \zeta(x, y, t)$ is shown to be arbitrary, with the proviso that ζ shall not vary so rapidly that extra curvature terms have to be included in the boundary layer equations, and the conditions for this to hold are investigated. It might be anticipated that difficulties would arise in attempts to apply the theorem when $\zeta < 0$, since in this case the new solution is not defined for $\zeta \leqslant z < 0$. However, a given original solution of the boundary layer equations can be extended for a certain distance across the surface $z = 0$ by analytic continuation, and it will be found that in practical applications ζ is only a small fraction of the boundary layer thickness. Numerical results will be obtained in terms of values and derivatives of the dependent variables at $z = 0$ in the original solution, and thus the theorem will be of use for both positive and negative values of ζ.

Fields of application of the theorem include flow past a rotating cylinder, slip flow, flow over a body covered in whole or part by a moving belt or a liquid film, and flow over a body of liquid. Some consideration is given to all these cases in this paper.

2. The Theorem

The boundary layer equations for unsteady compressible three-dimensional laminar flow can be written as follows, in terms of orthogonal curvilinear coordinates (x, y, z), where $z = 0$ is the body surface:

$$\left.\begin{aligned}
&\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{h_1 h_2}\frac{\partial(h_2 u)}{\partial x} + \frac{1}{h_1 h_2}\frac{\partial(h_1 v)}{\partial y} + \frac{\partial w}{\partial z} = 0,\\
&\rho\frac{Du}{Dt} + \frac{\rho uv}{h_1 h_2}\frac{\partial h_1}{\partial y} - \frac{\rho v^2}{h_1 h_2}\frac{\partial h_2}{\partial x} = -\frac{1}{h_1}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right),\\
&\rho\frac{Dv}{Dt} - \frac{\rho u^2}{h_1 h_2}\frac{\partial h_1}{\partial y} + \frac{\rho uv}{h_1 h_2}\frac{\partial h_2}{\partial x} = -\frac{1}{h_2}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right),\\
&\rho c_p\frac{DT}{Dt} = \frac{Dp}{Dt} + \mu\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right\} + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right).
\end{aligned}\right\} \quad (1)$$

Here $h_1\,dx$ and $h_2\,dy$ are elements of length in the coordinate directions, h_1 and h_2 being functions of x and y, (u, v, w) are the velocity components, μ and k are the coefficients of viscosity and thermal conductivity, and t is the time. The pressure p, the density ρ and the temperature T are also connected by the equation of state. The pressure does not vary across the boundary layer, and is known from conditions in the external flow.

Suppose one solution of these equations, Solution 1, is available, where the variables take values $(u_1, v_1, w_1, \rho_1, T_1)$ which are known functions of (x, y, z, t). They will satisfy certain conditions at the surface $z = 0$, and at the outer edge of the boundary layer, which may be considered to be at $z = \infty$ as far as the boundary layer equations are concerned, all except

w_1 will take prescribed values. Consider now a second set of expressions $(u_2, v_2, w_2, \rho_2, T_2)$ with the properties

$$\left.\begin{aligned} u_2(x,y,z,t) &= u_1(x,y,z+\zeta,t), \\ v_2(x,y,z,t) &= v_1(x,y,z+\zeta,t), \\ \rho_2(x,y,z,t) &= \rho_1(x,y,z+\zeta,t), \\ T_2(x,y,z,t) &= T_1(x,y,z+\zeta,t), \\ w_2(x,y,z,t) &= \left\{w_1 - \frac{\partial\zeta}{\partial t} - \frac{u_1}{h_1}\frac{\partial\zeta}{\partial x} - \frac{v_1}{h_2}\frac{\partial\zeta}{\partial y}\right\}_{x,y,z+\zeta,t}, \end{aligned}\right\} \tag{2}$$

where $\zeta = \zeta(x,y,t)$. Our theorem consists of the assertion that the quantities (2) also satisfy the equations (1), and so give a second solution, Solution 2, of the boundary layer equations. At the outer edge of the boundary layer, Solution 2 will satisfy the same conditions as Solution 1. The value of w will be different, but this is not a quantity which can be prescribed in a boundary layer problem. At the surface, $z = 0$, Solution 2 will satisfy quite different conditions from Solution 1, and it remains to be investigated whether they will be appropriate in any practical application.

The general form of the expressions (2) is in accord with the concept, mentioned in § 1, of taking a new surface at $z = \zeta$ in Solution 1. The extra contribution $-\partial\zeta/\partial t$ to w_2 results from the relative motion of the old and new boundaries, and the contribution $-(u_1/h_1)\partial\zeta/\partial x - (v_1/h_2)\partial\zeta/\partial y$ arises since u_1 and v_1 have components normal to the new surface. In boundary layer flow, w is small compared with u and v, so there are no corresponding additions to u_2 and v_2.

However, the truth of the theorem is most concisely and conclusively demonstrated by direct substitution in the boundary layer equations (1). From equations (2),

$$\frac{\partial u_2}{\partial z} = \frac{\partial u_1}{\partial z}, \tag{3}$$

$$\begin{aligned} \frac{Du_2}{Dt} &= \frac{\partial u_2}{\partial t} + \frac{u_2}{h_1}\frac{\partial u_2}{\partial x} + \frac{v_2}{h_2}\frac{\partial u_2}{\partial y} + w_2\frac{\partial u_2}{\partial z} \\ &= \left(\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial z}\frac{\partial\zeta}{\partial t}\right) + \frac{u_1}{h_1}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial z}\frac{\partial\zeta}{\partial x}\right) + \frac{v_1}{h_2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_1}{\partial z}\frac{\partial\zeta}{\partial y}\right) + \\ &\qquad + \left(w_1 - \frac{\partial\zeta}{\partial t} - \frac{u_1}{h_1}\frac{\partial\zeta}{\partial x} - \frac{v_1}{h_2}\frac{\partial\zeta}{\partial y}\right)\frac{\partial u_1}{\partial z} \\ &= \frac{\partial u_1}{\partial t} + \frac{u_1}{h_1}\frac{\partial u_1}{\partial x} + \frac{v_1}{h_2}\frac{\partial u_1}{\partial y} + w_1\frac{\partial u_1}{dz} = \frac{Du_1}{Dt}, \end{aligned} \tag{4}$$

and similarly for v_2, ρ_2, and T_2. Also,

$$\begin{aligned} \frac{1}{h_1h_2}\frac{\partial(h_2u_2)}{\partial x} + \frac{1}{h_1h_2}\frac{\partial(h_1v_2)}{\partial y} + \frac{\partial w_2}{\partial z} &= \frac{1}{h_1h_2}\left\{\frac{\partial(h_2u_1)}{\partial x} + h_2\frac{\partial u_1}{\partial z}\frac{\partial\zeta}{\partial x}\right\} + \\ + \frac{1}{h_1h_2}\left\{\frac{\partial(h_1v_1)}{\partial y} + h_1\frac{\partial v_1}{\partial z}\frac{\partial\zeta}{\partial y}\right\} &+ \left\{\frac{\partial w_1}{\partial z} - \frac{1}{h_1}\frac{\partial u_1}{\partial z}\frac{\partial\zeta}{\partial x} - \frac{1}{h_2}\frac{\partial v_1}{\partial z}\frac{\partial\zeta}{\partial y}\right\} \\ &= \frac{1}{h_1h_2}\frac{\partial(h_2u_1)}{\partial x} + \frac{1}{h_1h_2}\frac{\partial(h_1v_1)}{\partial y} + \frac{\partial w_1}{\partial z}. \end{aligned} \tag{5}$$

It is thus clear that equations (1) and the equation of state are satisfied, since p, h_1 and h_2 are not affected, and so the theorem is proved.

We should expect there to be some limitations on ζ for the theorem to hold. In boundary layer flow, $\partial^2 u/\partial z^2$ is $O(\delta^{-2})$, and no other second derivative of u exceeds $O(\delta^{-1})$, where δ is the thickness of the boundary layer, and the dimensions of the body are considered to be $O(1)$. This must continue to apply, or extra viscous terms will have to be included in equations (1). Now $\partial^2 u_2/\partial x^2$ has terms

$$(\partial^2 u_1/\partial z^2)(\partial\zeta/\partial x)^2 \quad \text{and} \quad (\partial u_1/\partial z)(\partial^2\zeta/\partial x^2),$$

and hence we must have

$$\frac{\partial\zeta}{\partial x} = O(\delta^{1/2}), \qquad \frac{\partial^2\zeta}{\partial x^2} = O(1). \tag{6}$$

Consideration of other derivatives leads to similar conclusions. Clearly $\zeta = O(\delta)$, or Solution 2 will lie entirely outside the original boundary layer. We see, therefore, that the typical distance for a change in ζ must not be less than $O(\delta^{1/2})$, i.e. a distance intermediate between the length of the body and the thickness of the boundary layer.

The direct physical interpretation of the theorem, as given by equations (2), is that a given solution may be displaced through a distance $-\zeta$ in the z-direction, without affecting its validity. This is not exactly the same as our previous idea of taking the body surface to be at $z = \zeta$ instead of at $z = 0$, as in the latter case a change of coordinate system is involved, the new coordinates being along and perpendicular to the new boundary. However, it would seem that on the boundary layer approximation the two physical processes are indistinguishable. If the change of coordinate system is not to affect the values given in equations (2), and if the curvature of the boundary is to be such that the solution is applicable to bodies of small or zero surface curvature, it can readily be shown that ζ must satisfy the conditions of equation (6), precisely as before. This gives further confirmation of the equivalence of the two physical interpretations of the theorem.

3. Rotating cylinder

Practical applications of the theorem can be found when ζ is a small fraction of the boundary layer thickness δ. For the simple case of steady two-dimensional incompressible flow, we may take $h_1 = 1$, and equations (2) reduce to

$$u_2(x, z) = u_1(x, z+\zeta), \qquad w_2(x, z) = w_1(x, z+\zeta) - u_1(x, z+\zeta)\frac{d\zeta}{dx}, \tag{7}$$

where $\zeta = \zeta(x)$. In terms of the stream function ψ, equations (7) are equivalent to

$$\psi_2(x, z) = \psi_1(x, z+\zeta), \tag{8}$$

in which form the theorem was noted by Prandtl (1938). By use of Taylor's theorem, we now obtain

$$\left.\begin{aligned} \psi_2 &= \psi_1 + \zeta\frac{\partial\psi_1}{\partial z} + \tfrac{1}{2}\zeta^2\frac{\partial^2\psi_1}{\partial z^2} + \dots \\ &= \psi_1 + \zeta u_1 + \tfrac{1}{2}\zeta^2\frac{\partial u_1}{\partial z} + \dots, \\ u_2 &= u_1 + \zeta\frac{\partial u_1}{\partial z} + \tfrac{1}{2}\zeta^2\frac{\partial^2 u_1}{\partial z^2} + \dots, \\ \frac{\partial u_2}{\partial z} &= \frac{\partial u_1}{\partial z} + \zeta\frac{\partial^2 u_1}{\partial z^2} + \dots . \end{aligned}\right\} \tag{9}$$

At $z = 0$, Solution 1 will normally satisfy $\psi_1 = u_1 = 0$. Also

$$\left(\frac{\partial u_1}{\partial z}\right)_{z=0} = \frac{\tau_1}{\mu}, \tag{10}$$

where τ is the skin-friction, and

$$\left(\frac{\partial^2 u_1}{\partial z^2}\right)_{z=0} = \frac{1}{\mu}\frac{dp}{dx} = -\frac{\rho U}{\mu}\frac{dU}{dx}, \tag{11}$$

from the boundary layer equations, where $U(x)$ is the velocity outside the boundary layer. Thus if $(\zeta/\delta)^2$ is negligible compared with unity, the conditions satisfied by Solution 2 at $z = 0$ are

$$\psi_2 = 0, \qquad u_2 = \zeta\tau_1/\mu, \qquad \tau_2 = \tau_1 - \zeta\rho U(dU/dx). \tag{12}$$

Consider the flow past a circular cylinder, rotating so that the surface has a small tangential velocity a. If Solution 1 gives the flow with zero velocity at the cylinder surface, then by (12) we obtain the required Solution 2 by choosing

$$\zeta = a\mu/\tau_1, \tag{13}$$

and hence

$$\tau_2 = \tau_1 - \frac{a\mu\rho U}{\tau_1}\frac{dU}{dx}. \tag{14}$$

Thus if τ_1 has been determined, perhaps by means of a Kármán–Pohlhausen calculation, τ_2 is given at once. It should be noted that the external flow $U(x)$, used in (14) and to obtain τ_1, must be that round the cylinder when it is rotating, and the determination of this flow is not a purely boundary layer problem. The condition for (12) to hold, that ζ/δ is small, implies that a/U must be small. This is clear physically, since the new surface in the Solution 1 flow must be at the point at which $u_1 = a$. As discussed in § 1, equation (14) applies whether a is positive or negative.

Separation and stagnation points

According to equation (13), ζ becomes large when τ_1 becomes small, and so our solution ceases to be valid, whatever the value of a. This will occur in the region of the front stagnation point and at separation. The question of what constitutes separation on a rotating cylinder is one of some complexity, which we shall not embark on here. We note that, according to (14), τ_2 becomes infinite when $\tau_1 = 0$. However, if we retain the terms in ζ^2 in (9), the boundary condition on u is satisfied if

$$a\mu = \zeta\tau_1 - \tfrac{1}{2}\zeta^2\rho U(dU/dx), \tag{15}$$

and with ζ chosen to satisfy (15) we obtain

$$\tau_2 = \{\tau_1^2 - 2a\mu\rho U(dU/dx)\}^{1/2}. \tag{16}$$

This expression for τ_2 is equivalent to (14) when τ_1 is not small, but does not become large near separation. If a is small, ζ is small for all values of τ_1, and the value of ψ_2 at $z = 0$ never exceeds $O(a/U)^{3/2}$. Thus (16) is applicable even near the separation point in Solution 1.

Consider the cylinder to be rotating so that the upper surface is moving in the same direction as the stream. Then a is positive on the upper surface and, according to equation (16), τ_2 has a finite value when $\tau_1 = 0$ (since dU/dx is negative here). The validity of Solution 1 ceases at this point, and so no further deduction of Solution 2 can be made, but presumably τ_2 falls steadily to zero as x increases further. On the lower surface a is negative, if x is still measured in the flow direction, and $\tau_2 = 0$ when $\tau_1^2 = 2a\mu\rho U(dU/dx)$. The solution cannot be extended further by use of the theorem, as there is no point in Solution 1, or its continuation below $z = 0$, which could be taken as the new position of the surface. In any case, it is natural to consider τ_2 to be zero beyond this point.

Near the front stagnation point, where $U < a$, there is no point in the boundary layer at which $u_1 = a$, and so the theorem must cease to apply. However, the value of τ_2 given by equation (14) has a finite limit at the stagnation point, and we can show that this value is correct. Consider the linear approximation to (9),

$$\psi_2 = \psi_1 + \zeta u_1, \qquad u_2 = u_1 + \zeta(\partial u_1/\partial z). \tag{17}$$

If $\zeta(\partial u_1/\partial z)_{z=0} = a$, the boundary conditions at $z = 0$ are satisfied exactly, and the boundary layer equations are satisfied if terms quadratic in ζ are neglected. Thus

$$u_2 = u_1 + a\left(\frac{\partial u_1}{\partial z}\right) \Big/ \left(\frac{\partial u_1}{\partial z}\right)_{z=0}. \tag{18}$$

In this form it is clear that, in the boundary layer equations, the terms in a^2 remain finite as $x \to 0$, and will be negligible if a is small, and so the equations are adequately satisfied. Equation (14) follows as before, and so is valid even in the neighbourhood of the stagnation point. Incidentally, Rott (1956) and Glauert (1956) have shown that, for a linear external velocity distribution $U = cx$, equation (18) gives the exact flow for a surface velocity a of any magnitude.

Torque on cylinder

We now attempt to make a quantitative estimate of the torque G on a slowly rotating cylinder. The first thing to be decided is the form of the velocity distribution $U(x)$ at the edge of the boundary layer. Experiments by Reid and by Betz quoted by Goldstein (1938, p. 546) show that no significant lift is observed if $a < \frac{1}{2}U_0$, where U_0 is the velocity of the oncoming stream, and so we shall assume that $U(x)$ remains the same as when the cylinder is not rotating. In accordance with the values given by Hiemenz (Goldstein 1938, p. 150), we choose

$$U/U_0 = 1{\cdot}8138\theta - 0{\cdot}2705\theta^3 - 0{\cdot}04708\theta^5, \tag{19}$$

where θ is the angular coordinate round the cylinder. For a non-rotating cylinder, the skin-friction is now given by the series due to Howarth (1935), with additional coefficients calculated by Ulrich (1949), as

$$R^{1/2}\frac{\tau_1}{\rho U_0^2} = 4{\cdot}258\theta - 1{\cdot}493\theta^3 - 0{\cdot}287\theta^5 + 0{\cdot}0190\theta^7 - 0{\cdot}0004\theta^9, \tag{20}$$

where the Reynolds number $R = U_0 d/\nu$, d being the cylinder diameter and ν the kinematic viscosity.

On the basis of (19) and (20), the value of $(\tau_2 - \tau_1)/a$ as given by equation (14) can be tabulated as a function of θ, and the results are shown in figure 1. Over the front part of the cylinder, the skin-friction produces a decelerating torque, tending to reduce the rotation, but beyond the velocity maximum there is an accelerating contribution to G/a, which according to (14) is logarithmically infinite. There is thus a tendency to autorotation.

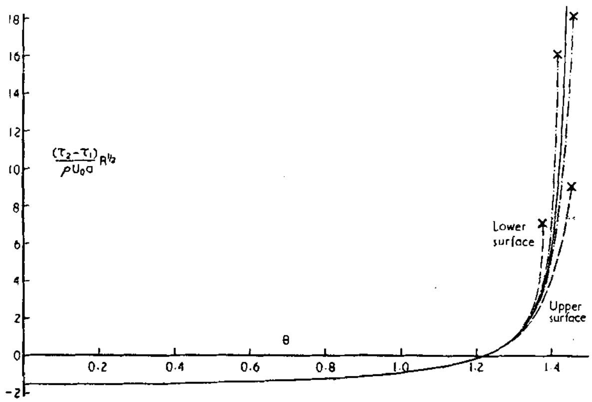

Figure 1. Increment of skin-friction $(\tau_2 - \tau_1)$ on a circular cylinder rotating with surface velocity a in a stream U_0, at an angle θ from the front of the cylinder. ——— a/U_0 very small (equation (14)), ——— · ——— · $a/U_0 = 0{\cdot}025$, —— —— —— $a/U_0 = 0{\cdot}1$, x limit of validity.

However, this result is valid only for vanishingly small a; for larger values of a, equation (16) must be used in place of equation (14), in the separation region. The values given by (16) cannot be represented by a single curve for all values of a. In figure 2 the values of τ_2 are shown directly for $a/U_0 = 0{\cdot}025$ and $a/U_0 = 0{\cdot}1$. Here equation (14) has been used to obtain the values up to the velocity maximum (where $\tau_2 = \tau_1$), and equation (16) from there on. The corresponding values of $(\tau_2 - \tau_1)/a$ are also shown in figure 1.

We now calculate the torque by integrating the skin-friction over the cylinder, as far as the zero of τ_2 on the lower surface and the zero of τ_1 on the upper. As discussed above, these are the limits of validity of our solution. We obtain the following results.

$$\left.\begin{aligned} &\text{When} \quad \frac{a}{U_0} = 0{\cdot}025, \quad R^{1/2}\frac{G}{\rho U_0^2 d^2} = -0{\cdot}0088. \\ &\text{When} \quad \frac{a}{U_0} = 0{\cdot}1, \quad R^{1/2}\frac{G}{\rho U_0^2 d^2} = -0{\cdot}0475. \end{aligned}\right\} \tag{21}$$

The negative sign indicates that the torque G acts to decrease the rotation. Certain sources of error in the values of equation (21) may be noted. On the upper surface, τ_2 must fall to zero over a certain distance beyond the point at which $\tau_1 = 0$, and this will give an additional accelerating torque. However, it would appear from figure 2 that the final values of τ_2 are probably overestimates, due to the increasing inaccuracy of our approximate solution, and also there may be a significant torque over the separated region at the rear of the cylinder. These two effects will produce an additional deceleration. Further errors will have arisen from our choice of a symmetrical velocity distribution, and in sum there seems little justification for modifying the values of equation (21), though no great confidence can be placed in their accuracy.

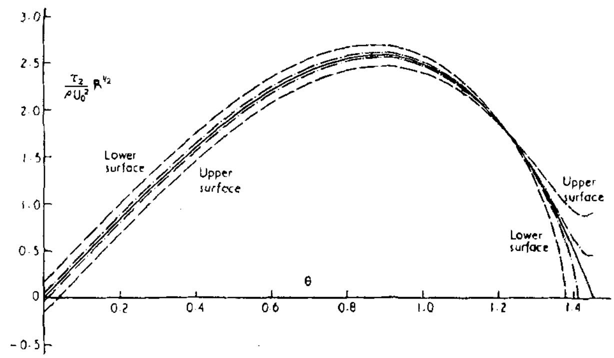

Figure 2. Skin friction τ_2 on a circular cylinder rotating with surface velocity a in a stream U_0, at an angle θ from the front of the cylinder.
———— $a/U_0 = 0$ (no rotation); —— · —— · $a/U_0 = 0{\cdot}025$;
—— —— —— $a/U_0 = 0{\cdot}1$.

Finally, a rough calculation shows that if a/U_0 is less than $0{\cdot}001$, G is positive and so tends to produce autorotation. On the basis of this estimate, a cylinder falling freely with velocity U_0 should rotate, with a surface velocity $0{\cdot}001\,U_0$.

For a cylinder making rotational oscillations with frequency Ω, our solution will continue to apply provided $\partial\zeta/\partial t$ is not too large, so that, from equation (2), the boundary condition on w_2 is adequately satisfied. It is easily shown that this requires that $\Omega d/U_0$ shall be small compared with unity. For a cylinder making translational oscillations (Glauert 1956), the torque is expressible as a power series in $\Omega d/U_0$. No doubt the same is true in the present case, the theorem enabling us to estimate the first term of the series.

4. Slip flow

For the high speed flow of a gas of low density, the boundary conditions at the surface may be modified to permit a slip velocity and a temperature jump. In the notation of § 2, the boundary conditions at $z = 0$ for three-dimensional flow are

$$\left.\begin{aligned} u &= L\,\partial u/\partial z, \\ v &= L\,\partial v/\partial z, \\ w &= 0, \\ T - T_W &= L^*\,\partial T/\partial z, \end{aligned}\right\} \tag{22}$$

where T_W is the temperature of the solid surface and L and L^* are lengths of the order of the molecular mean free path. The basis of these conditions is discussed by Nonweiler (1952), who gives an extensive treatment of the two-dimensional case, using what is in effect a form of our theorem for small ζ. We shall show briefly how the theorem may be applied in the general three-dimensional case, as was attempted by Mangler (1956), who, however, omitted all curvature terms from the boundary layer equations.

Use of Taylor's theorem in equation (2) shows that, if $(\zeta/\delta)^2$ is negligible and the flow is steady,

$$\left.\begin{aligned} u_2 &= u_1 + \zeta\,\partial u_1/\partial z, \\ v_2 &= v_1 + \zeta\,\partial v_1/\partial z, \\ T_2 &= T_1 + \zeta\,\partial T_1/\partial z, \\ w_2 &= w_1 + \zeta\frac{\partial w_1}{\partial z} - \frac{u_1}{h_1}\frac{\partial \zeta}{\partial x} - \frac{v_1}{h_2}\frac{\partial \zeta}{\partial y}. \end{aligned}\right\} \tag{23}$$

Solution 1 will normally satisfy $u_1 = v_1 = w_1 = 0$ at $z = 0$. The conditions (22) on u, v and w will then be satisfied by choosing

$$\zeta = L, \tag{24}$$

since, from the first of equations (1), $\partial w_1/\partial z = 0$ when $z = 0$. For the temperature, it follows from (22), (23) and (24) that, at $z = 0$,

$$T_W = T_1 + (L - L^*)\,\partial T_1/\partial z. \tag{25}$$

The following expressions for τ_x and τ_y, the components of the skin-friction, and $q = (k\,\partial T/\partial z)_{z=0}$, the heat-transfer rate at the surface, are now readily obtained with the use of Taylor's theorem and the values of equations (1) at $z = 0$.

$$\left.\begin{aligned} (\tau_x)_2 &= (\tau_x)_1 + \frac{L}{h_1}\frac{\partial p}{\partial x}, \\ (\tau_y)_2 &= (\tau_y)_1 + \frac{L}{h_2}\frac{\partial p}{\partial y}, \end{aligned}\right\} \tag{26}$$

$$q_2 = q_1 - L\mu\left\{\left(\frac{\partial u_1}{\partial z}\right)^2 + \left(\frac{\partial v_1}{\partial z}\right)^2\right\}. \tag{27}$$

Thus, corresponding to a known Solution 1, we can find a Solution 2 applicable to slip flow, the variation of temperature and heat transfer over the surface being given by equations (25) and (27). The extra contributions to the skin-friction are given by equations (26), without even requiring a knowledge of the Solution 1 values. There are no difficulties either at a rounded nose or at separation, since L/δ remains small. At a sharp leading edge the solution ceases to be valid, but so do the boundary layer equations themselves.

For two-dimensional flow, all these results were obtained by Nonweiler, although his expression for w_2 was incorrect. On the assumption that the flow is incompressible, Lin & Schaaf (1951) gave the results for the special cases of flow over a flat plate and near the stagnation point on a body of revolution.

Difficulties occur in practice. If the wall temperature is to be constant, or the rate of heat transfer is to be zero, it is insufficient to employ a Solution 1 with the same property. As shown by (25) and (27), the required Solution 1 must satisfy a slightly modified temperature condition, and hence $(\tau_x)_1$ and $(\tau_y)_1$ will differ from their values in the corresponding no-slip case by terms proportional to L. Nonweiler makes several attempts to estimate the effects of such modifications.

5. Further applications

Brief consideration will now be given to certain other classes of problem in which the theorem might have useful applications. In all cases the velocity distribution outside the boundary layer must be specified, and this may be difficult in some instances.

For the flow past a body covered in whole or part by a moving belt, the theorem is applicable if the belt velocity is small compared with the stream velocity, the treatment following that of § 3. As is clear from equation (23) the theorem gives a solution only if the belt moves in the same direction as the fluid near the surface, so this application is effectively restricted to two-dimensional motions. If the belt extends over only part of the body, difficulties arise when considering the flow near its ends, as ζ would have to change more rapidly than permitted by the theorem.

For the flow past a body covered with a liquid film, Solution 1 would be computed assuming the film to be frozen. The skin-friction induces a motion in the film, and the effect of this motion on the flow over the body could be found from Solution 2. Since the liquid moves in the direction of the stress, the theorem will be applicable for a general three-dimensional body.

For the flow past a body of liquid, Solution 1 would be calculated assuming the body to be solid. If the internal motion of the liquid could then be estimated, its effect on the air boundary layer would be given by the theorem. Here the applications will usually be confined to two-dimensional or axi-symmetric motions.

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